Abstract—Inverse optimal control is a classical problem of control theory. It was first posed by Kalman in the early sixties. The problem, as addressed in literature, answers to the following two questions: (a) Given system matrices $A, B$ and a gain matrix $K$, find necessary and sufficient conditions for $K$ to be the optimal of an infinite time LQ problem. (b) Determine all weight matrices $Q, R$ and $S$ which yield the given gain matrix $K$. In this paper, we tackle a related, but different problem. Starting from the state trajectories of an LTI system, identify the matrices $Q, R$ and $S$ that have generated those trajectories. Both infinite and finite time optimal control problems are considered. The motivation lies in the characterization of the trajectories of LTI systems in terms of the control task.

I. INTRODUCTION

Inverse optimal control theory dates back to the early sixties. The theory was inspired by a problem first posed by Kalman [1]. Consider the classical infinite time LQ direct optimal control:

$$
\min_u \int_{t_0}^{\infty} \left[ x^\top(t) Q x(t) + u^\top(t) R u(t) + 2x^\top(t) S u(t) \right] \, dt
$$

s.t. \begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
 x(t_0) &= x_0
\end{align*}

$x \in \mathbb{R}^n, u \in \mathbb{R}^m$.

(1)

Under general hypothesis ($Q = Q^\top, R = R^\top > 0$ and $(A, B)$ stabilizable), it can be proven [2] that if an optimal control $u^*$ exists, then it corresponds to a static state feedback $K$, i.e. $u^*(t) = K x(t)$. The inverse problem considered by Kalman consists in two questions:

(a) Find necessary and sufficient conditions on the matrices $A, B$ and $K$ such that the control law $u(t) = K x(t)$ minimizes the cost (1) for some $Q, R$ and $S$.

(b) Determine all such costs, i.e. all $Q, R$ and $S$ corresponding to the same $K$.

A. Previous works on inverse optimal control

In this section we report previous results on the solution of (a) and (b). The underlying hypothesis are the same of the direct problem plus a condition on $B$: $\text{rank}(B) = m$. Moreover, (1) is slightly modified by adding a stabilizing constraint:

$$
\lim_{t \to \infty} x(t) = 0.
$$

Kreindler [3] solved (a) showing that $K$ is optimal for some $Q = Q^\top, R = R^\top > 0$ and $S$ if and only if $A + BK$ is asymptotically stable. Moreover, he showed that (b) can be solved arbitrarily fixing $Q = Q^\top$ and $R = R^\top > 0$, and then computing a suitable $S$.

Jameson [4] restricted the set of cost functions, imposing $S = 0$. Under this restriction, $K$ is optimal if and only if $A + BK$ is asymptotically stable and $KB$ has $m$ linearly independent real eigenvectors. All $Q$ and $R$ that lead to the same $K$ are determined solving in $P = P^\top$ and $R = R^\top$ the matrix equation $B^\top P = -R K P$. $Q$ is then computed using the standard algebraic Riccati equation, with $Q$ as the unknown. The obtained results are easily generalized to the time-varying finite-horizon LQ optimal control problem.

Molinari [5] considered an even more restrictive situation, fixing $R = I_m$ and $S = 0$. Under those restraints, $K$ is optimal if and only if $A + BK$ is asymptotically stable and $KB$ is symmetric. The set of all $Q = Q^\top$ leading to a given $K$ can be obtained solving:

$$
Q = \tilde{Q} + A^\top Y + YA,
$$

where $Y = Y^\top$ is any solution of $Y B = 0$ and where $\tilde{Q}$ is any weighting matrix leading to the given $K$.

The solution of (a) considerably changes if $Q$ is required to be positive semidefinite. Specifically, the request $Q = Q^\top \geq 0$ leads to the so called “return difference condition”. Its single input formulation is due to Kalman [1] and was later extended by Anderson and Moore [6]. Specifically, $K$ is optimal for some $Q = Q^\top \geq 0, R = R^\top, S = 0$ if and only if $A + BK$ is stable and there exists an $R = R^\top > 0$ that satisfy the return difference inequality:

$$
[I + B^\top (-j \omega I - A)^{-1} K] R
\geq
[I + K(j \omega I - A)^{-1} B] \quad \forall \omega \in \mathbb{R}.
$$

(2)

B. Cost estimation problem

In this paper we address a problem related to inverse optimal control. Specifically, we consider the problem of estimating a cost function that best approximate a given set of state trajectories. The underlying hypothesis is that the given trajectories result from the optimal control of a known linear system at different initial conditions. We aim at estimating the...
Consider the following optimal control problem:
\[
\min_u \frac{1}{2} \int_{t_0}^{\infty} J(x(t), u(t)) dt \quad \text{s.t.} \quad \begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0 \end{cases}
\]
where \((A, B)\) is supposed stabilizable and the matrix \(B\) full column rank. Let the cost function contain an input-state cross term:
\[
J(x, u) = x^T Q x + u^T R u + 2 x^T S u.
\]  \(\text{(3)}\)

From now on, we will indicate \(\mathcal{J}\) the set of such cost functions with standard hypothesis on the matrices \(Q, R\) and \(S\), i.e.:
\[
\mathcal{J} = \left\{ J: J \text{ of type (3) with } Q = Q^T \geq 0, R = R^T > 0, \right\}
\]
the matrices \(\tilde{A}, \tilde{Q}\) are defined as follows:
\[
\tilde{A} = A - BR^{-1}S^T, \quad \tilde{Q} = Q - SR^{-1}S^T.
\]  \(\text{(5)}\)

Under these hypothesis the solution of the optimal control problem is known to be unique and corresponds to an asymptotically stabilizing static state feedback \(K\), i.e. \(u^*(t) = Kx(t)\) with \(A + BK\) stable. Let \(x^*(t; x_0)\) be the optimal trajectory, i.e. the state trajectory corresponding to the optimizing input \(u^*(t)\), when the initial condition is \(x(t_0) = x_0\). Then we formulate the following problem.

**Problem 1:** Assume that we are given the matrices \(A\) and \(B\) and a set of optimal trajectories for different initial conditions:
\[
\{x_i(t) = x^*(t; x_{0,i}) : i = 1, \ldots, N\}.
\]
Estimate the cost function \(J \in \mathcal{J}\) (or equivalently the matrices \(Q, R\) and \(S\)) that have generated the given trajectories.

The problem does not have a unique solution, because optimal trajectories do not univocally determine a cost function in \(\mathcal{J}\). Specifically, previous works have shown that for any cost function in the set \(\mathcal{J}\) there exists an infinite number of different (not simply scaled) cost functions leading to the same optimal gain \(K\). Therefore, given a set of optimal trajectories with different initial conditions, we cannot univocally determine the matrices \(Q, R, S\) that generated those trajectories. The following definition will help us in formalizing this concept.

**Definition 1:** \((J \sim \tilde{J}, \text{equivalence relation on } \mathcal{J})\). Consider \(J, \tilde{J} \in \mathcal{J}\). We say that \(J\) is equivalent to \(\tilde{J}\) and we write \(J \sim \tilde{J}\) if and only if \(K = \tilde{K}\), being \(K\) and \(\tilde{K}\) the optimal gains associated to \(J\) and \(\tilde{J}\), respectively.

It can be shown that the relation \(\sim\) defined on the set of the cost functions \(\mathcal{J}\) is an equivalence, being reflexive, symmetric and transitive. Consequently, we can divide \(\mathcal{J}\) into equivalence classes.

Coming back to our problem, a given set of trajectories generated by \(J\), could have been generated by any other \(\tilde{J}\) in the equivalence class associated to \(J\). Therefore, in order to define correctly the estimation problem it is necessary to identify a canonical representative in each class; afterwards, we can then reduce the problem to estimating the canonical form of \(J\). The problem can be reformulated as follows.

**Problem 2:** Assume that we are given the matrices \(A\) and \(B\) and a set of optimal trajectories for different initial conditions:
\[
\{x_i(t) = x^*(t; x_{0,i}) : i = 1, \ldots, N\}.
\]
Estimate the canonical form of the equivalence class associated to \(J \in \mathcal{J}\), being \(J\) the cost function that have generated the given trajectories.

Let us then build a set of canonical forms, i.e. a set that contains exactly one element in each equivalence class. Consider the following set of cost functions:
\[
J_1(x, u) = x^T K^T K x + u^T u - 2 x^T K^T u, \quad J_1 \in \mathcal{J}_1 = \left\{ J_1 \text{ of type (6)} \right\}
\]  \(\text{(6)}\)

**Proposition 1:** \(J_1\) is a proper subset of the set \(\mathcal{J}\), i.e. \(\mathcal{J}_1 \subset \mathcal{J}\).

**Proof:** Consider \(J_1 \in \mathcal{J}_1\); we want to prove that \(J_1 \in \mathcal{J}\). Equivalently, we have to show that \(Q \triangleq K^T K, \ R \triangleq I_m\) and \(S \triangleq -K^T\) satisfy the properties in (4). Obviously \(K^T K \geq 0\) and \(I_m > 0\). Moreover:
\[
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq \begin{bmatrix} K^T K & -K^T \\ -K & I_m \end{bmatrix} \geq 0
\]
as follows from the semidefinite positiveness of the Schur complement \(Q - SR^{-1}S^T = 0 \geq 0\) and from the positiveness of the matrix \(R\). It remains to prove that \((\tilde{A}, \tilde{Q})\) is detectable. Easy substitution in (5) leads to:
\[
\tilde{A} = A + BK, \quad \tilde{Q} = 0.
\]
Using (7) we conclude \(\lambda(\tilde{A}) < 0\) and the detectability of the couple \((\tilde{A}, \tilde{Q})\) easily follows. It remains to prove that \(\exists J_1 \in \mathcal{J}\) such that \(J_1 \notin \mathcal{J}_1\); this trivially follows from the definitions.  

**Proposition 2:** Consider \(J_1 \in \mathcal{J}_1\). If:
\[
J_1(x, u) = x^T K^T K x + u^T u - 2 x^T K^T u,
\]
then the optimal gain associated to \(J_1\) is exactly \(K\).

**Proof:** Using proposition 1 we have that \(J_1 \in \mathcal{J}\) so that the associated direct optimal control problem can be solved in the standard manner. Specifically, define \(Q \triangleq K^T K, \ R \triangleq I_m\) and \(S \triangleq -K^T\). The optimal gain associated to \(J_1\) is given by:
\[
K_1 = -R^{-1}(B^T P + S^T),
\]
where $P$ is the unique positive semidefinite solution of the following Algebraic Riccati Equation:

$$P\dot{A} + \dot{A}^T P - PBR^{-1}B^T P + Q = 0.$$  \hspace{1cm} (8)

Easy substitutions show that $Q = 0$ so that $P = 0$. Substituting those results in the expression of the optimal gain we obtain $K_1 = -R^{-1}S^T = K$, and this concludes the proof.

**Proposition 3:** The set $J_1$ is a set of canonical forms for the equivalence relation $\sim$ defined on $J$.

**Proof:** Since we have already proven that $J_1 \subset J$, we are left with proving that the set $\bar{J}_1$ contains exactly one element in each equivalence class. We proceed by proving that for every $J \in \bar{J}_1$ there is at least one element $J_1 \in \bar{J}_1$ such that $J \sim J_1$; we then prove that this element is unique. Consider $J \in \bar{J}_1$ and let $K$ be the optimal gain associated to $J$; using the hypothesis on $J$ we know that $\lambda(A + BK) < 0$ (see [6] for details). Consequently, the following cost function:

$$J_1(x, u) = x^T K^T K x + u^T u - 2x^T K^T u.$$  

is a cost function in $\bar{J}_1$. Moreover, such a $J_1$ is in the equivalence class of $J$, i.e. $J \sim J_1$. In fact, using proposition 2, the optimal gain $K_1$ associated to $J_1$ equals $K$ and this concludes the first part of the proof. Let’s now prove that there are not two elements of $\bar{J}_1$ in the same equivalence class, i.e. there are not in $\bar{J}_1$ two elements equivalent to one another. Suppose by contradiction that there exist $J_1$ and $\bar{J}_1 \in \bar{J}_1$ such that $J_1 \sim \bar{J}_1$ and $J_1 \neq \bar{J}_1$. Therefore:

$$J_1(x, u) = x^T K^T K x + u^T u - 2x^T K^T u,$$

$$\bar{J}_1(x, u) = x^T \bar{K}^T \bar{K} x + u^T u - 2x^T \bar{K}^T u.$$

If $J_1 \neq \bar{J}_1$, then $K \neq \bar{K}$. But $K$ and $\bar{K}$ are the optimal gains associated to $J_1$ and $\bar{J}_1$ respectively and consequently the optimal gains associated to $J_1$ and $\bar{J}_1$ are different. This contradicts the assumption $J_1 \sim \bar{J}_1$ and concludes the proof.

**Proposition 4:** Consider the system $\dot{x} = Ax + Bu$ and let $x(t)$ be a state trajectory. Then, there exists a cost function $J \in \bar{J}$ that has $x(t)$ as minimizing trajectory if and only if there exists a stabilizing gain $K$ such that:

$$\dot{x}(t) = (A + BK)x(t) \quad \forall t.$$  \hspace{1cm} (9)

**Proof:** ($\Rightarrow$) It easily follows from the theory of optimal control. ($\Leftarrow$) Take:

$$J(x, u) = x^T K^T K x + u^T u - 2x^T K^T u,$$

and use proposition 2 to show that the optimal trajectory is exactly (9).

At this point problem 2 can be formulated as a constrained parametric identification problem; the model will be the following:

$$\begin{cases} \dot{x}(t) = (A + BK)x(t) + v(t) \\ y(t) = x(t) + w(t) \end{cases}$$

where everything is known but the noise variances $\Sigma_v$, $\Sigma_w$, and the matrix $K$, constrained to be stabilizing for $(A,B)$. Measurements are the trajectories $x^*(t; x_{0,i}, x_{1,i})$ truncated after a sufficiently long time interval. The main obstacle in solving this identification problem consists in forcing the matrix $K$ to be stabilizing.

**III. Cost Function Estimation: Finite Time Optimal Control with Fixed Final State**

In this section we consider a finite time optimal control with fixed final state. To our knowledge, the associated inverse optimal control problem has never been considered in literature.

$$\min_u \frac{1}{2} \int_{t_0}^{t_1} J(x(t), u(t))dt$$

s.t. $\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0, \quad x(t_1) = x_1 \end{cases}$

Let’s make the standard assumption on the cost function $J$, i.e. let’s assume $J \in \bar{J}$; moreover let the couple $(A,B)$ be controllable and the matrix $B$ be full column rank. Under these assumptions the direct problem solution is well known. However, in this paper we do not refer to the classical expression for the optimal control $u^*(t)$ but to a reformulation recently proposed by A. Ferrante et al. (see [7] for details).

Specifically, it can be proven that:

$$u^*(t) = -K_+e^{A_+t}p_1 - K_-e^{-A_-(t-t_f)}p_2$$

where the quantities $K_+, K_-, A_+, A_-, p_1$ and $p_2$ can be determined as follows:

$$A_+ = A + BK_+,$$

$$K_+ = -R^{-1}(S^T + B^TP),$$

$$K_- = K_+ + R^{-1}B^T \Delta,$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} I_n \\ e^{A_+t_f} \\ \cdots \\ I_n \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

The matrix $P$ is the unique stabilizing and positive semidefinite solution of the Algebraic Riccati Equation (8); $\Delta$ is the solution of the following Lyapunov equation:

$$A_- \Delta^{-1} + \Delta^{-1}A_+^T + BR^{-1}B^T = 0.$$

Using the well known properties of the Lyapunov equation and the stability of the matrix $A_+$, we know that $\Delta$ is uniquely determined as:

$$\Delta = \left( \int_0^\infty e^{A_-t}BR^{-1}B^Te^{A_+t}dt \right)^{-1}.$$

At this point, a problem similar to the one presented in the previous section can be formulated. In this case, trajectories are assumed to be optimal with respect to an LQ fixed final state problem with different initial and final conditions.

**Problem 3:** Assume we are given the matrices $A$ and $B$ and a set of optimal trajectories for different initial and final conditions:

$$\{ x_i(t) = x^*(t; x_{0,i}, x_{1,i}) : i = 1, \ldots, N \}.$$

Estimate the cost function $J \in \bar{J}$ (or equivalently the matrices $Q$, $R$ and $S$) that have generated the given trajectories.
Also in this case the problem does not have a unique solution. Specifically, one can easily find different (not simply scaled) cost functions, $J$ and $J'$, that lead to the same $K_+$ and $K_-$. Therefore the trajectories that follow from the minimization of $J$ will be equal to the ones that follow from the minimization of $J'$, for any initial and final condition. Consequently, the following definition of equivalence is the natural modification of the definition seen in the Section II.

**Definition 2:** ($J \sim \bar{J}$, equivalence relation on $\mathcal{J}$). Consider $J, \bar{J} \in \mathcal{J}$. We say that $J$ is equivalent to $\bar{J}$ and we write $J \sim \bar{J}$ if and only if $K_+ = K_+$ and $K_- = K_-$. 

Once again, it can be shown that the relation $\sim$ defined on the set of the cost functions $\mathcal{J}$ is an equivalence and consequently $\mathcal{J}$ can be divided into equivalence classes. Notice that a given set of optimal trajectories generated by $J$, could have been generated by any other $\bar{J}$ in the equivalence class associated to $J$. Ideally, one would like to associate a canonical form to each equivalence class. The set of canonical forms should contain exactly one element for each equivalence class. However, up to this moment, we haven’t found such a set; instead, in the following we will introduce a set $\mathcal{J}_2 \subset \mathcal{J}$ that contains at least one element in equivalence class:

$$J_2(x, u) = x^T K_+ R K_+ x + u^T R u - 2 x^T K_+ R u$$

where we have defined $\bar{R} \triangleq \frac{R}{\det(R)}$. Let’s show that $\bar{J}$ is effectivelly in $\mathcal{J}_2$, i.e. let’s show that $\lambda(A + BK_+) < 0$ and that $\bar{R}$ is unitary and positive definite. $\lambda(A + BK_+) < 0$ from the fact that $A + BK_+ = A_+$ is asymptotically stable; the second is evident from the definition of $\bar{R}$ and from the positiveness of $R$. To prove that $J \sim \bar{J}$ we have to show that $K_+$ and $K_-$ equal respectively $K_+$ and $K_-$. Starting from $K_+$ and using $J_2 \in \mathcal{J}$ we have:

$$P \tilde{A} + (\tilde{A}^T P - PBR^{-1} B^T P + \tilde{Q}) = 0,$$

$$K_+ = -R^{-1}(B^T P + S^T),$$

where $P$ is the unique stabilizing and positive semidefinite solution of (12) and:

$$\bar{Q} \triangleq K_+^T R K_+,$$

$$\bar{A} \triangleq A - BR^{-1} S^T,$$

$$\bar{Q} \triangleq \bar{A} - \bar{Q}^{-1} S^T$$

and let $K_+$ and $K_-$ be the matrices associated to $J$ through the solution of the associated optimal control problem. An element $\bar{J} \in \mathcal{J}_2$ equivalent to $J$ is the following:

$$\bar{J}(x, u) = x^T K_+ R K_+ x + u^T \bar{R} u - 2 x^T K_+ \bar{R} u.$$

**Proof:** Consider $J_2 \in \mathcal{J}_2$: we want to show that $J_2 \in \mathcal{J}$. We have to show that $Q \triangleq K_+^T R K_+$ and $S \triangleq -K_+^T R$ satisfy the properties in (4). Obviously, $R = R^T > 0$ by (11) and therefore $Q = K_+^T R K_+ > 0$. Moreover:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} K_+^T R & -K_+^T R \\ -K_+^T R & R \end{bmatrix} \geq 0,$$

as follows from the semidefinite positiveness of the Schur complement $Q - S R^{-1} S^T = 0 \geq 0$ and from the positiveness of the matrix $R$. It remains to prove that $(\tilde{A}, \tilde{Q}^{1/2})$ is detectable. Easy substitution leads to:

$$\tilde{A} = A + BK, \quad \tilde{Q} = 0,$$

so that the detectability follows from the fact that $\lambda(A + BK) < 0$ as a consequence of $J_2 \in \mathcal{J}_2$. It remains to prove that $\exists J \in \mathcal{J}$ such that $J \notin \mathcal{J}_2$: this trivially follows from the definitions.

**Proposition 6:** The set $\mathcal{J}_2$ contains at least one element in each equivalence class in $\mathcal{J}$.

**Proof:** We have to prove that for every $J \in \mathcal{J}$ there is at least one element $\bar{J} \in \mathcal{J}_2$ such that $J \sim \bar{J}$. Let:

$$J(x, u) = x^T Q x + u^T R u + 2 x^T S u,$$

and let $\lambda(t) = \int_{t_0}^{t_1} e^{A_+ t} B R^{-1} B^T e^{A_+ t} dt$.

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} A + BK & -BR^{-1} B^T \\ 0 & -A^T - K_+ B^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

**Proof:** To prove what we claim we start from a well known result in linear quadratic convex optimal control (see [8] for details). It essentially states that if $J \in \mathcal{J}$ then a
trajectory \( x(t) \) is optimal for the associated fixed final state optimal problem -with initial and final condition respectively \( x(t_0) \) and \( x(t_1) \)- if and only if the following condition holds:

\[
\exists \lambda(t) : \begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} \hat{A} & -BR^{-1}B^T \\ -\hat{Q} & -\Lambda^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix},
\]

(15)

where we defined as usual:

\[
\hat{A} \triangleq A - BR^{-1}S^T, \quad \hat{Q} \triangleq Q - SR^{-1}S^T.
\]

Let’s now prove proposition 7. (Only if) Consider \( x(t) \), optimal for \( J \in \mathcal{J} \). Then, according to Proposition 6, it must be optimal for some \( \bar{J} \in \mathcal{J}_2 \):

\[
\bar{J}(x,u) = x^T K^T R K x + u^T R u - 2x^T K^T R u.
\]

(16)

Since \( \mathcal{J}_2 \subset \mathcal{J} \), condition (15) must hold with \( Q = K^T R K, \ S = -K^T R \) and \( R = R^T > 0 \) unitary; then, substitutions in (15) lead to (14) and the result is proven. (If) Suppose that (14) holds for a \( K \) (stabilizing), \( R \) (unitary and positive definite) and \( \lambda(t) \); we prove that \( x(t) \) is optimal for \( \bar{J} \) defined as in (16) with the given \( K \) and \( R \). The result easily follows observing that (15) corresponds to (14) defining \( Q = K^T R K \), and \( S = -K^T R \).

Problem 3 becomes a mixed identification-estimation problem. Specifically we have:

\[
\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A + BK & -BR^{-1}B^T \\ O_n & -(A + BK)^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + v(t)
\]

\[ y(t) = x(t) + w(t) \]

where the measurements are the optimal trajectories \( x_i(t) = x^*(t; x_{i0}, x_{1,i}) \) for \( i = 1, \ldots, N \). The variables to be estimated are the matrices \( K \) and \( R \) together with the state variable \( \lambda(t) \). The estimation procedure should constrain the unknown matrices \( K \) and \( R \) to be respectively stabilizing for the couple \( (A, B) \) and positive definite.

IV. Conclusions

This paper was about the problem of estimating a cost function from a given set of state trajectories; matrices \( A \) and \( B \) of the underlying linear system were assumed known. Two classes of optimal control problems were considered: infinite time optimal control and finite time optimal control with fixed final state; in both cases, cost functions were assumed to be quadratic. The estimation problem turned out to be ill-posed by the non-uniqueness of the matrices \( Q, R \) and \( S \) that correspond to the same optimal control. This complication was handled by defining a proper equivalence relation on the set of admissible cost functions and searching for a set of canonical forms. The problem of estimating the cost function was then reduced to the constrained identification of a linear “gray-box” state space model.